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BOUNDARY LAYER GROWTH ON CONTINUOUSLY FORMED SURFACES

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Boundary layers adjacent to continuous surfaces moving with constant velocity from a stationary source represent a class of boundary layer phenomena often encountered in practice, for example, processes involving the rolling and extrusion of materials, and atomization processes where liquid initially in the form of cylindrical jets or sheets are ejected from a nozzle. Generally, in the latter case, the coherent liquid stream persists only for a brief time, typically less than a few milliseconds, but during this period, the nature of the boundary layer may play a significant role in establishing the mechanism by which the stream breaks down into drops. In particular, it has been shown (Crapper *et al.* 1973) to influence the growth of aerodynamic waves on rapidly moving liquid sheets, and to affect the onset of electrohydrodynamic wave motion (Clark & Dombrowski 1974) by controlling the diffusion of charged species from a hot ionized gaseous environment.

The first account of work on this subject appears to be that due to Howarth (1959) who examined the general case of laminar boundary layer flow adjacent to a solid of revolution. Subsequently, Sakiadis (1961a, b) published a similar, but more detailed study of the boundary layer flow adjacent to a plane surface. He approached the problem in two ways; the first of these involved the numerical solution of the boundary layer equations, whilst the second was concerned with obtaining an analytical solution by the Polhausen integral method. The results showed that although the latter could describe the surface drag with reasonable accuracy, the velocity profile, and the parameters relating to entrainment and boundary layer thickness, were subject to more significant errors. In this note, we present an alternative solution which more accurately approaches the numerical one over the entire flow field.

It is easily demonstrated (Meksyn 1961) that the equations describing boundary flow adjacent to a solid of revolution reduce to a form similar to that for two-dimensional flow,

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and thus, following Howarth we may, without loss of generality, seek a solution of the Blasius equation,

$$f''' + ff'' = 0 \qquad [1]$$

which satisfies the boundary conditions imposed by a moving continuous surface, namely,

$$f' = 1, \quad f = 0 \quad \text{at} \quad \eta = 0, \quad \text{and} \quad f' \to 0 \quad \text{as} \quad y \to \infty.$$

For the solution scheme presented here we take as the starting point the integral equation,

$$f' = 1 + a \int_0^{\eta} \exp(-F(\eta))\, d\eta \qquad [2]$$

where $F(\eta) = \int_0^{\eta} f\, d\eta$ and a is a constant of integration (i.e. $a = f''(0)$). The integrand of equation [2] can be seen to be a rapidly decreasing function with a stationary point at $\eta = 0$, and we may therefore obtain a solution by the method of steepest descent (Meksyn 1961).

Let

$$f = \eta + \sum_2^{\infty} \frac{a_n \eta^n}{!n}$$

where f satisfies the boundary conditions at $\eta = 0$. Substituting the expression into [1] we obtain the coefficients a_n in terms of $a_2 = a$ (see equation [2]).

$$f = \eta + \frac{a}{!2}\eta^2 - \frac{a}{!4}\eta^4 - \frac{a^2}{!5}\eta^5 + \frac{3a}{!6}\eta^6 + \frac{11a^2}{!7}\eta^7 \ldots \qquad [3]$$

The unknown parameter, a, is found from [2] and the boundary condition at infinity, viz.

$$1 + a \int_0^{\infty} \exp(-F(\eta))\, d\eta = 0 \qquad [4]$$

and is evaluated by means of the following substitutions

$$\eta = \sum_0^{\infty} (A_m/(m+1))\tau^{(m+1)/2} \qquad [5]$$

and

$$F(\eta) = \eta^2 \sum_0^{\infty} C_n \eta^n = \tau \qquad [6]$$

where by comparison with [3],

$$C_0 = \tfrac{1}{2}, \quad C_1 = a/!3, \quad C_2 = 0, \quad C_3 = -a/!5$$
$$C_4 = -a^2/!6, \quad C_5 = 3a/!7, \quad C_6 = 11a^2/!8, \ldots .$$

Combining [4], [5] and [6],

$$a \int_0^{\infty} e^{-\tau} \sum_0^{\infty} \frac{A_m}{2} \tau^{(m-1)/2}\, d\tau = -1 \qquad [7]$$

which on integration in gamma-functions gives

$$1 + \frac{a}{2}\left(\sum_0^\infty A_m \Gamma\left(\frac{m+1}{2}\right)\right) = 0. \qquad [8]$$

In order to evaluate the coefficients A_m we differentiate [5] to give

$$d\eta = \sum_0^\infty \frac{A_m}{2} \tau^{(m-1)/2}\, d\tau$$

whence it follows that,

$$\int^{0+} \frac{d\eta}{\tau^{(m+1)/2}} = \frac{A_m}{2} \int^{0+0+} \frac{d\tau}{\tau} = 2i\pi A_m \qquad [9]$$

where 0+ denotes a circuit in a positive direction around the zero point, and the single circuit around $\eta = 0$ corresponds to a double circuit round $\tau = 0$ in the plane. This integration path is necessary to dispose of the fractional powers of τ. It follows, therefore, from [9] that A_m is the coefficient of η^{-1} in the expansion of $\tau^{-(m+1)/2}$ in ascending powers of η.

Now from [6].

$$\tau^{-(m+1)/2} = \eta^{-(m+1)}(C_0 + C_1\eta + C_3\eta^3 \ldots)^{-(m+1)/2} \qquad [10]$$

and expanding the expression by the binomial expansion gives,

$$A_0 = C_0^{-1/2}, \quad A_1 = \frac{C_1}{C_0} C_0^{-1}, \quad A_2 = \frac{15}{8}\left(\frac{C_1}{C_0}\right)^2 C_0^{-3/2},$$

$$A_3 = -C_0^{-2}\left(2\frac{C_3}{C_0} + 4\left(\frac{C_1}{C_0}\right)^3\right)$$

$$A_4 = C_0^{-5/2}\left(-\frac{5}{2}\frac{C_4}{C_0} + \frac{35}{4}\frac{C_3C_1}{C_0^2} + \frac{3465}{384}\left(\frac{C_1}{C_0}\right)^4\right)$$

$$A_5 = C_0^{-3}\left(-3\frac{C_5}{C_0} + 12\frac{C_1C_4}{C_0^2} - 30\frac{C_1^2C_3}{C_0^3} - 21\left(\frac{C_1}{C_0}\right)^5\right).$$

The parameter, a, has been evaluated from [8] by substituting for the coefficients, A_m, and solving the generated power series by successive approximation. Convergence was achieved with the first six terms of the series giving a value of a of -0.6265.

Table 1. Numerical values of coefficients A_m

A_0	1.4142
A_1	0.4176
A_2	0.2312
A_3	0.0619
A_4	0.0046
A_5	−0.0028

Table 2. Tabulated values of the functions f' and η

τ	η	$\sqrt{2\eta}$	f'
0	0	0	1.0
0.2	0.6805	0.9624	0.5997
0.3	0.8514	1.2041	0.5198
0.4	1.001	1.4156	0.4513
0.5	1.1360	1.6065	0.3968
0.6	1.2628	1.7859	0.3510
0.7	1.3830	1.9559	0.3116
0.8	1.4980	2.1185	0.2784
0.9	1.6093	2.2759	0.2477
1.0	1.7173	2.4286	0.2212
2.0	2.7017	3.8208	0.0751
3.0	3.6351	5.1408	0.0270
4.0	4.5659	6.4572	0.0090

This may be compared with the exact value of -0.6275 obtained by Howarth (1959) and Sakiadis (1961b). The first six coefficients, A_m, are listed in table 1. The variation of velocity across the boundary layer can now be obtained directly from [2] after integrating in incomplete gamma-functions, viz.

$$f' = 1 + \frac{a}{2}\sum_{0}^{\infty} A_m \Gamma\left(\tau, \frac{m+1}{2}\right) \tag{11}$$

where η and τ are related by [5].

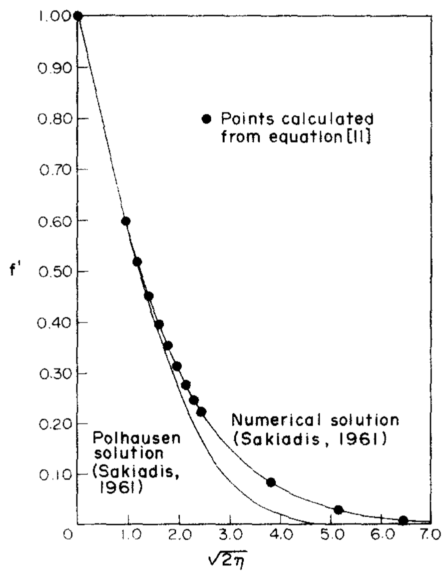

Figure 1. Comparison of numerical and analytical solutions.

Table 3. Comparison of values of boundary layer coefficients determined by numerical and analytical methods

	Numerical solution (Sakiadis 1961b)	Polhausen solution (Sakiadis 1961b)	Present solution
Displacement thickness	1.143	0.990	1.152
Momentum thickness	0.6275	0.605	0.6265

Some values of the functions f' and η are listed in table 2 for a range of the variable τ. They have been derived using values of the incomplete gamma-functions tabulated by Pearson (1922). These results show excellent agreement with the numerical solution,† and this is illustrated in figure 1, together with a curve derived by Sakiadis using a Polhausen integral method.

Sakiadis (1961b) has also calculated approximate and exact values of two further boundary layer parameters, namely, the displacement thickness (δ^*) and momentum thickness (θ). Using the above form of velocity profile, we obtain

$$\delta^* = 1.152\left(\frac{2\nu x}{u}\right)^{1/2} \quad [12]$$

$$\theta = 0.6265\left(\frac{2\nu x}{u}\right)^{1/2}. \quad [13]$$

A comparison is made in table 3 between the boundary layer coefficients given by equations [12] and [13] and the corresponding values presented by Sakiadis. It is seen that the present solution gives the momentum and displacement thickness to within 1% of their exact values.

CONCLUSIONS

The work has demonstrated that the method of steepest descent for solving boundary layer equations can be applied more accurately than the Polhausen integral method to the problem of continuously formed surfaces. Excellent agreement between the analytical and numerical solutions has been obtained for the velocity distribution and the displacement and momentum thicknesses.

†Equation [1] is identical to that presented by Sakiadis (1961b) for plane flow except that, in the latter's derivation, a constant equal to 2 appears in the coefficient of the term ff''. This arose from the less conventional choice of the definition of η as $y(u/\nu x)^{1/2}$ rather than $\eta = y(u/2\nu x)^{1/2}$. His solutions may therefore be applied to the general case after correcting for the factor $\sqrt{2}$.

REFERENCES

CLARK, C. J. & DOMBROWSKI, N. 1974 An experimental study of the flow of thin liquid sheets in hot atmospheres. *J. Fluid Mech.* **64**, 167–175.

CRAPPER, G. D., DOMBROWSKI, N., JEPSON, W. P. & PYOTT, G. A. D. 1973 A note on the growth of Kelvin-Helmholtz waves on thin liquid sheets. *J. Fluid Mech.* **57**, 671–672.

HOWARTH, L. 1959 Appendix to a paper by G. I. Taylor, The dynamics of thin sheets of fluid, 1. Water bells. *Proc. Roy. Soc.* **A253**, 289–295.

MEKSYN, D. 1961 *New Methods in Laminar Boundary Layer Theory*. Pergamon Press.

PEARSON, K. 1922 *Tables of the Incomplete Gamma-Function*. H.M.S.O.

SAKIADIS, B. C. 1961a Boundary layer behaviour on continuous solid surfaces, 1. Boundary layer equations for two-dimensional and axi-symmetric flow. *A.I.Ch.E.Jl.* **7**, 26–28.

SAKIADIS, B. C. 1961b Boundary layer behaviour on continuous solid surfaces, 2. The boundary layer on a continuous flat surface. *A.I.Ch.E.Jl.* **7**, 221–225.